

ROM2F/2011/06

MULTI INSTANTON TESTS OF HOLOGRAPHY

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*Dipartimento di Fisica, Università di Roma “Tor Vergata”**Via della Ricerca Scientifica, 00133 Roma, Italy***Abstract**

Gauge theories living on stacks of D7-branes are holographically related to IIB gravitational backgrounds with a varying axion-dilaton field (F-theory). The axion-dilaton field is generated by D7, O7 and D-instanton sources and can be written in terms of the chiral correlators of the eight dimensional gauge theory living on the D7-branes. Using localization techniques, we prove that the same correlators determine the gauge coupling of the four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theories living on the elementary D3-brane which probes the F-theory geometries.

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1 Introduction and Summary

F-theory provides us with some of the few known explicit examples of non-conformal gauge gravity duals at the non-perturbative level. The simplest setting involves a stack of N D7-branes on top of an O7-plane in flat ten-dimensional spacetime¹. The D7 worldvolume theory is described by a maximally supersymmetric gauge theory in eight space time dimensions with gauge group $SO(2N)$. The gravity side is given by F-theory at a D_N -singularity or equivalently by a type IIB orientifold with varying axion-dilaton field, $\tau(z)$, with z the coordinate on the complex plane transverse to the D7 branes.

The $SO(8)$ case, with $N = 4$ D7-branes on top of the O7-plane, is special because the tadpoles are exactly canceled and the axion-dilaton field, τ_0 , is constant along the z -plane. Taking the D7-branes away from the O7 plane, let us say at positions a_u , generates a non trivial dependence $\tau(z, \tau_0, a_u)$ for the axion-dilaton field. Relying on the $SO(8)$ symmetries of the background, it was proposed in [1] to identify this function with the gauge coupling $\tau_{\text{gauge}}(z, \tau_0, m_u)$ described by the Seiberg-Witten curve of a $\mathcal{N} = 2$

¹This local brane system arises in type I' theory (the theory obtained by T-dualizing a T^2 torus of type I theory on T^2) after locating N D7-branes in the neighborhood of one of the four O7 planes.

supersymmetric $SU(2)$ gauge theory with four hypermultiplets of masses $m_u = a_u$, transforming in the fundamental representation of the gauge group [2]. z parametrizes the Coulomb branch of the moduli space of the gauge theory and τ_0 is the UV coupling. This proposal was further supported by the observation in [3] that the same gauge theory describes the dynamics of a D3-brane probe of the F-theory geometry.

More recently in [4], the $SO(8)$ F-theory background was explicitly derived from string diagrams describing the rate of emission of the axion-dilaton field from D7, O7 and D(-1) sources. The results were written in the suggestive form [5]

$$\tau_{\text{sugra}}(z, \tau_0, a) = \langle \mathcal{O}_\tau(\Phi) \rangle_{D7} \quad a = \langle \Phi \rangle_{D7} \quad (1.1)$$

where

$$\mathcal{O}_\tau(a) = \tau_0 + \ln \det \left(1 - \frac{a}{z} \right) \quad (1.2)$$

is the operator in the eight space time dimensional gauge theory dual to the axion-dilaton field on the gravity side. a is an 8×8 antisymmetric matrix with purely imaginary eigenvalues $\pm a_u$ parametrizing the D7-brane positions. The operator \mathcal{O}_τ is determined by disk amplitudes involving an insertion of a τ -field and any number of open string vertices. We remark that the correlator on the r.h.s. of (1.1) receives an infinite tower of D(-1)-instanton corrections resulting into an infinite series in $e^{2\pi i \tau_0}$ for the axion-dilaton field.

Holography requires that the same function describes the gauge coupling τ_{gauge} on the four space time dimensional theory living on the D3-brane probe

$$\tau_{\text{gauge}}(z, \tau_0, a) = \langle \mathcal{O}_\tau(\Phi) \rangle_{D7} \quad (1.3)$$

This is consistent with the fact that, according to (1.2), $\mathcal{O}_\tau(a)$ is nothing but the perturbative contribution (tree and one-loop) to the gauge coupling on the D3-brane probe. D-instantons modify both sides of (1.3) in a rather asymmetric way and therefore an agreement at the non-perturbative level is far from obvious. In particular, exotic instantons in the eight dimensional theory (the right hand side) are entangled with the “standard” gauge instantons in four space time dimensions (the left hand side).

The intriguing relation (1.3) was verified in [5] for the first few instanton orders by an explicit comparison of the results coming from the corresponding Seiberg-Witten curve against those for the eight space time dimensional chiral correlators computed in [6]. The aim of this paper is to prove that this

relation holds at any instanton order. The proof will rely on multi instanton formulae obtained via localization [7, 8]. The analysis will be extended to a large class of F-theory backgrounds with sixteen and eight supersymmetric charges. More precisely, we will consider systems made of N D7 branes and an O7 plane on \mathbb{R}^{10} or $\mathbb{R}^6 \times \mathbb{R}^4/\mathbb{Z}_2$. The associated gauge symmetries of the D7-worldvolume theories are $SO(2N)$ and $U(N)$ respectively. Both eight-dimensional backgrounds will be tested via elementary D3-branes supporting a $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theories with N fundamentals. The instanton moduli spaces for these brane systems were worked out in [6, 9, 10].

Finally we remark that the relation (1.3) holds even in the extreme case $N = 0$, where no D7-branes are included in the F-theory background. The τ_{sugra} in this case corresponds to the axion-dilaton field generated by a single O7-plane. On the other hand τ_{gauge} describes the gauge coupling of the pure $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory living on a D3-brane probe of this geometry. We stress that even in this simple case one finds a non-trivial F-theory elliptic fibration incorporating the effects of D-instantons on the geometry. This case can be thought of as the decoupling limit where the masses of all the fundamentals are sent to infinity by bringing the D7-branes far away from the O7 plane. More interestingly, the correlator on the right hand side of (1.1) describes well defined F-theory geometries also in the case with $N > 4$ D7-branes where the $SU(2)$ gauge theory in the probe is not asymptotically free and a Seiberg-Witten type analysis is not available. It would be nice to understand what is the geometry underlying such F-theory backgrounds.

2 $SO(2N)$ models

We start by considering a system of D(-1), D3 and D7 branes in presence of an O7 plane in a flat ten dimensional space. The D(-1)-branes or D-instantons can be thought of either as gauge instantons on the D3-brane or as exotic instantons on the eight dimensional gauge theory living on the D7-branes. In each case, the instanton moduli are associated to the massless modes of the open strings connecting the various branes.

The background is specified by the positions, a_u , of the D7-branes in the transverse plane which parametrize the vevs of the chiral scalar field Φ in the eight dimensional worldvolume theory. More precisely, using the $SO(2N)$ invariance one can always bring the antisymmetric matrix $\langle \Phi \rangle_{\text{cl}}$ into the

diagonal form

$$\langle \Phi \rangle_{\text{cl}} = \text{diag}\{a_u, -a_u\} \quad (2.4)$$

with a_u purely imaginary, $u = 1, \dots, N$. At the quantum level the background is modified by the presence of the D-instantons and it is better described by the eight dimensional prepotential $\mathcal{F}(\Phi)$ and the chiral correlators $\langle \text{tr} \Phi^J \rangle_{D7}$. The prepotential and chiral correlators are computed by integrals over the multi-instanton moduli space [6, 9]. After suitable deformations that regularize the eight dimensional spacetime volume $\text{vol}_{D7} \sim \frac{1}{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}$, these integrals localize around a finite number of points where they can be explicitly evaluated. More precisely, the integrals over the multi-instanton moduli space reduce to contour integrals on χ_i (the positions of the D(-1)-branes) of a super-determinant $\det Q^2$, with Q an equivariantly deformed BRST charge. The parameters $\epsilon_\ell, a_u, \chi_i$ together with the positions b_m of the D3-brane probes parametrize the Lorentz and gauge symmetries broken by the instantons and determine the eigenvalues of Q^2 .

2.1 The D(-1)D7 system

We start by considering the D(-1)D7 system with no D3-branes. The theory on the D7-branes lives in eight dimensions and carries a gauge group $SO(2N)$. Instantons are realized by the inclusion of K fractional D(-1)-branes with symmetry group $SO(K)$. In the following we will use the notations employed in [6] to which we refer for further details.

The K -instanton partition function of the D7 worldvolume theory can be written as

$$Z_{D7,K} = \int d\mathcal{M}_K e^{-S_{\text{mod}}(\mathcal{M}_K)} = \int \prod_{i=1}^k \frac{d\chi_i}{2\pi i} w_{D7}(\chi, a) \quad (2.5)$$

with $w_{D7}(\chi)$ the contributions of D(-1)D(-1) and D(-1)D7 open strings to the determinant of Q^2 . The matrix χ is $K \times K$ antisymmetric with purely imaginary eigenvalues χ_I specifying the positions of the instantons and their images along the plane transverse to D7-branes. We write

$$\chi_I = \begin{cases} (\chi_i, -\chi_i) & \text{for } SO(2k) \\ (\chi_i, -\chi_i, 0) & \text{for } SO(2k+1) \end{cases} \quad i = 1, \dots, k \quad (2.6)$$

depending on whether K is even or odd. We denote by $k = [\frac{K}{2}]$ the number of regular instantons with positions χ_i . We notice that for K odd a fractional instanton is always stuck at the origin. The integral over χ_i can be

computed as a contour integral on the upper half plane after taking the poles prescription $1 \gg \text{Im}\epsilon_1 \gg \text{Im}\epsilon_2 \gg \text{Im}\epsilon_3 \gg \text{Im}\epsilon_4 \gg \text{Im}a_u$. In addition we require

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0 \quad (2.7)$$

in order to ensure the invariance of the BRST charge Q . The integrand in (2.5) can be written as [6]

$$\begin{aligned} w_{D7}(\chi, a) = & \frac{c_K}{k!} \prod_{\ell=1}^4 \prod_{I,J}^K \left[\frac{(\chi_{IJ} + s_\ell)}{(\chi_{IJ} + \epsilon_\ell)} \right]^{\frac{1}{2}} \prod_{\ell=1}^4 \prod_{I=1}^K \left[\frac{1}{(2\chi_I + \epsilon_\ell)(2\chi_I + s_\ell)} \right]^{\frac{1}{2}} \\ & \times \prod_{I=1}^K \prod_{u=1}^N (\chi_I - a_u) \end{aligned} \quad (2.8)$$

with $\chi_{IJ} = \chi_I - \chi_J$ and

$$s_1 = 0 \quad s_2 = \epsilon_1 + \epsilon_2 \quad s_3 = \epsilon_1 + \epsilon_3 \quad s_4 = \epsilon_2 + \epsilon_3 \quad (2.9)$$

The prime in (2.8) denotes the omission of the zero eigenvalues $(\chi_{II} + s_1)$ from the product. The three products in (2.8) correspond to the contributions of D(-1)D(-1), D(-1)O7 and D(-1)D7 open strings respectively. In particular the denominator of the first product comes from the four complex matrices B_ℓ , $\ell = 1, \dots, 4$ specifying the positions of the instantons along the D7 worldvolume while the numerator accounts for the generalized ADHM constraints D_ℓ defining the instanton moduli space. The second product implements the projections of B_ℓ and D_ℓ into the symmetric and adjoint representations of $SO(K)$ respectively. Finally the last product accounts for the D(-1)D7 strings with half the degrees of freedom of a bifundamental of $SO(K) \times SO(2N)$. Plugging (2.6) into (2.8) one can see that all square roots cancel out as expected. Finally $c_K = (-)^{k+1}(2)^{k-K}$ are numerical coefficients².

The prepotential is defined as

$$\mathcal{F}_{D7} = - \lim_{\epsilon_\ell \rightarrow 0} \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \ln Z_{D7}(q) \quad (2.10)$$

with

$$Z_{D7}(q) = \sum_{K=0}^{\infty} Z_{D7,K} q^K \quad (2.11)$$

²They can be determined requiring that \mathcal{F}_{D7} defined below is finite in the limit $\epsilon_\ell \rightarrow 0$.

and $Z_{D7,0} = 1$. The chiral correlators are given in terms of the same integrals with extra χ -insertions according to [6]

$$\langle \text{tr} e^{z\Phi} \rangle_{\text{inst}} = \frac{1}{Z_{D7}} \prod_{\ell=1}^4 (1 - e^{z\epsilon_\ell}) \sum_{K=1}^{\infty} q^K \int \prod_{i=1}^k \frac{d\chi_i}{2\pi i} w_{D7}(\chi, a) \text{tr} e^{z\chi} \quad (2.12)$$

In the limit $\epsilon_\ell \rightarrow 0$ (2.12) reduces to

$$\langle \text{tr} e^{z\Phi} \rangle_{\text{inst}} = \lim_{\epsilon_\ell \rightarrow 0} \frac{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}{Z_{D7}} \sum_{K=1}^{\infty} q^K \int \prod_{i=1}^k \frac{d\chi_i}{2\pi i} w_{D7}(\chi, a) z^4 \text{tr} e^{z\chi} \quad (2.13)$$

or equivalently

$$\left\langle \text{tr} \frac{\Phi^{J+4}}{(J+4)!} \right\rangle_{\text{inst}} = \lim_{\epsilon_\ell \rightarrow 0} \frac{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}{Z_{D7}} \sum_{K=1}^{\infty} q^K \int \prod_{i=1}^k \frac{d\chi_i}{2\pi i} w_{D7}(\chi, a) \text{tr} \frac{\chi^J}{J!} \quad (2.14)$$

with $\langle \text{tr} \Phi^J \rangle_{\text{inst}} = 0$ for $J < 4$. We remark that the correlator in the right hand side of (2.14) is defined even in the case $N = 0$ (no D7-branes) where the left hand side of this equation loses its sense. In this case, we will loosely keep the notation $\langle \text{tr} \Phi^J \rangle_{\text{inst}}$ as a shorthand for the associated correlator on the right hand side of (2.14).

2.2 The coupling on the D3-brane probe

Now we introduce a stack of M D3-branes probing the D7-background geometry. The theory living on the D3-branes is $\mathcal{N} = 2$ supersymmetric with gauge group $Sp(M)$ and flavor group $SO(2N)^3$. The $Sp(M)$ matter content is given by one hypermultiplet transforming in the antisymmetric representation and N hypermultiplets transforming in the fundamental representation of the gauge field.

The beta function coefficient reads ⁴

$$b_{Sp(M)} = 4 - N \quad (2.15)$$

³In our conventions $Sp(1) = SU(2)$.

⁴We use the conventions: $b = 2 \left[T(\text{adj}) - \sum_r n_r T(r) \right]$ with $T(r)$ the index of the representation r of $Sp(M)$ given by $T(\text{adj}) = \frac{2M+2}{2}$, $T(\square) = \frac{2M-2}{2}$ and $T(\square) = \frac{1}{2}$.

The case $N = 4$ is special in the sense that the four dimensional gauge theory is conformal for any M .

Now let us consider the effects of D-instantons. The K -instanton partition function is given by

$$Z_{D3,K} = \int \prod_{i=1}^k \frac{d\chi_i}{2\pi i} w_{D7}(\chi, a) w_{D3}(\chi, b) \quad (2.16)$$

with w_{D7} given by (2.8) and

$$w_{D3}(\chi, b) = \prod_{m=1}^M \prod_{I=1}^K \frac{(\chi_I - b_m)^2 - \left(\frac{\epsilon_3 - \epsilon_4}{2}\right)^2}{(\chi_I - b_m)^2 - \left(\frac{\epsilon_1 + \epsilon_2}{2}\right)^2} \quad (2.17)$$

giving the contributions of D(-1)D3 strings. The four dimensional prepotential is defined by [10]

$$\mathcal{F}_{D3}(a_u, b_m) = - \lim_{\epsilon_\ell \rightarrow 0} \epsilon_1 \epsilon_2 \ln \left(\frac{Z_{D3}(a_u, b_m)}{Z_{D7}(a_u)} \right) \quad (2.18)$$

The term $\epsilon_1 \epsilon_2 \ln Z_{D7} = -\frac{1}{\epsilon_3 \epsilon_4} \mathcal{F}_{D7}$ subtracts a b_m -independent divergent term of eight-dimensional origin and leaves a finite result for the four dimensional prepotential \mathcal{F}_{D3} .

The gauge coupling on the D3-probe is defined by

$$2\pi i \tau_{mn}^{\text{inst}} = \frac{\partial^2 \mathcal{F}_{D3}}{\partial b_m \partial b_n} \quad (2.19)$$

In the rest of this section we will show that τ_{mn} can be written entirely in terms of the chiral correlators of the D7 brane gauge theory .

First plugging (2.18) into (2.19) and using the fact that Z_{D7} does not depend on b_m one finds

$$2\pi i \tau_{mn}^{\text{inst}} = \lim_{\epsilon_\ell \rightarrow 0} \epsilon_1 \epsilon_2 \left[-\frac{1}{Z_{D3}} \frac{\partial^2 Z_{D3}}{\partial b_m \partial b_n} + \frac{1}{Z_{D3}^2} \frac{\partial Z_{D3}}{\partial b_m} \frac{\partial Z_{D3}}{\partial b_n} \right] \quad (2.20)$$

Now let us evaluate the right hand side of (2.20) in the limit $\epsilon_\ell \rightarrow 0$ limit. Taylor expanding w_{D3} in (2.17) in the previous limit and for large b_m , leads to

$$w_{D3}(\chi, b) = 1 + \epsilon_3 \epsilon_4 \sum_{J=0}^{\infty} \sum_{m=1}^M \frac{\text{tr} \chi^J}{b_m^{J+2}} (J+1) + \dots \quad (2.21)$$

where (2.7) was used. The behavior $w_{D3}(\chi, b) \approx 1$ follows from the fact that the D3-D3 sector is effectively $\mathcal{N} = 4$ supersymmetric and therefore has trivial instanton corrections to the four dimensional prepotential. The $\mathcal{N} = 4$ supersymmetry is broken by the D7-branes which introduce fundamental matter and by the O7-plane which projects the $\mathcal{N} = 2$ vector multiplet and the hypermultiplet into different representations of the gauge group. These contributions are encoded in the $w_{D7}(\chi, a)$ term in (2.8).

Plugging (2.21) into (2.16), one sees that only the first term in (2.20) contributes at order $\epsilon_3 \epsilon_4$ and one finds $\tau_{mn}^{\text{inst}} = \delta_{mn} \tau_m^{\text{inst}}$ with

$$\begin{aligned} 2\pi i \tau_m^{\text{inst}} &= - \lim_{\epsilon_\ell \rightarrow 0} \frac{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}{Z_{D7}} \sum_{K=1}^{\infty} \sum_{J=0}^{\infty} q^K \frac{(J+3)!}{J!} \int \prod_{i=1}^k \frac{d\chi_i}{2\pi i} w_{D7}(\chi, a) \frac{\text{tr} \chi^J}{b_m^{J+4}} \\ &= - \sum_{J=1}^{\infty} \frac{1}{J b_m^J} \langle \text{tr} \Phi^J \rangle_{\text{inst}} = \left\langle \ln \det \left(1 - \frac{\Phi}{b_m} \right) \right\rangle_{\text{inst}} \end{aligned} \quad (2.22)$$

The result (2.22) shows that the gauge coupling matrix of the $Sp(M)$ gauge theory factorizes, in the limit $\epsilon_\ell \rightarrow 0$, into M copies of a $Sp(1) \sim SU(2)$ gauge theories, each copy coming with N massive flavors transforming in the fundamental representation of $SU(2)$. Without losing generality we can then set $M = 1$ and rename $\tau_m \rightarrow \tau$, $b_m \rightarrow b$. Moreover, the result (2.22) shows that the gauge coupling τ can be written as an infinite sum of chiral correlators in the eight dimensional flavor gauge theory! The correlators $\langle \text{tr} \Phi^J \rangle_{\text{inst}}$ are given by the moduli space integrals (2.14). Explicit results up to $k = 3$ are collected in the Appendix.

2.2.1 Explicit results up to $k = 3$

In this section we display some explicit formulas for the first few instanton contributions to τ_{inst} . Plugging (A.50) into (2.22) one finds

$$\begin{aligned} \tau_{\text{inst}}(b) &= - \sum_{J=0}^{12} \frac{1}{J} \left\langle \text{tr} \frac{\Phi^J}{b^J} \right\rangle_{\text{inst}} \\ &= \frac{3q \sqrt{A_N}}{b^4} + q^2 \left(\frac{3A_{N-2}}{32b^4} - \frac{15A_{N-1}}{16b^6} + \frac{105A_N}{32b^8} \right) \\ &\quad + q^3 \sqrt{A_N} \left(\frac{3A_{N-4}}{64b^4} - \frac{5A_{N-3}}{16b^6} + \frac{35A_{N-2}}{32b^8} - \frac{21A_{N-1}}{8b^{10}} + \frac{165A_N}{32b^{12}} \right) \end{aligned} \quad (2.23)$$

with A_m , $m = 1, \dots, N$ a basis for the Casimirs of $SO(2N)$ given by

$$A_s = \sum_{i_1 < i_2 \dots < i_s} a_{i_1}^2 \dots a_{i_s}^2$$

$$A_N = a_1^2 \dots a_N^2 \quad A_0 = 1 \quad A_{s < 0} = 0 \quad (2.24)$$

Formula (2.23) determines the axion-dilaton field generated by a stack of N D7-branes. For $N \leq 4$ it matches the gauge coupling coming from the Seiberg-Witten curve associated to a four dimensional theory with gauge group $SU(2)$ and N fundamental hypermultiplets. Explicitly the curves for $N < 4$ can be written as

$$y^2 + R(x)y + q = 0 \quad (2.25)$$

with

$$R(x) = \frac{x^2 - e^2}{\sqrt{\prod_{u=1}^N (x - a_u)}} \quad (2.26)$$

and

$$\lambda = \frac{dz}{2\pi i} \frac{zR'(z)}{\sqrt{R(z)^2 - 4q}} \quad (2.27)$$

the Seiberg-Witten differential. b and $\frac{\partial \mathcal{F}}{\partial b}$ are defined by the two periods of the differential and $\tau = \frac{\partial^2 \mathcal{F}}{\partial b^2}$ ⁵. The conformal case $N = 4$ was worked out in [5].

We remark that the F-theory axion-dilaton result (2.23) is well defined even in cases like $N = 0$ where the eight-dimensional gauge theory is missing or $N > 4$ where the four-dimensional gauge theory on the D3-brane probe is not asymptotically free and curves of the Seiberg-Witten type are not available. The former case corresponds to an F-theory background made out of an O7-plane and no D7-branes. The eight-dimensional amplitude in this case follows from (2.23) after setting $N = 0$. We recall that in the case of no D7-branes the notation $\langle \text{tr} \Phi^J \rangle_{\text{inst}}$ is a shorthand for the D-instanton correlator on the right hand side of (2.14). Up to the third order in q one finds

$$\tau_{\text{inst}}(b) = - \sum_{J=0}^{12} \frac{1}{J} \left\langle \text{tr} \frac{\Phi^J}{b^J} \right\rangle_{\text{inst}} = 3 \frac{q}{b^4} + \frac{105}{32} \frac{q^2}{b^8} + \frac{165}{32} \frac{q^3}{b^{12}} \quad (2.28)$$

⁵Alternatively the prepotential from the Matone relation $2e^2 = 2q \frac{dF}{dq} + 2a^2$.

which reproduces the first instanton corrections to the gauge coupling of the pure $SU(2)$ gauge theory as claimed. The F-theory background generated by O7, D(-1) branes is then described by the pure $SU(2)$ Seiberg- Witten curve given by (2.25) with $N = 0$.

3 $U(N)$ models

The results we found in last section can be easily generalized to F-theory backgrounds with less supersymmetries and different gauge groups. Consider the same brane system on $\mathbb{R}^6 \times \mathbb{R}^4/\mathbb{Z}_2$. In general \mathbb{Z}_2 acts on the Chan-Paton indices and projects the brane symmetry groups into unitary or SO/Sp gauge groups for regular and fractional branes respectively. One can easily see that in the case of fractional branes the orbifold projects out the numerator of $w_{D3}(\chi, b)$ in (2.17) and therefore the condition $w_{D3} \approx 1$ cannot be realized in the limit $\epsilon_\ell \rightarrow 0$. This implies that a probe theory with $w_{D3} \approx 1$ can be engineered only for regular branes and therefore the reasonings following (2.21) only apply in that case.

We consider then a \mathbb{Z}_2 orbifold breaking $SO(2N) \times Sp(M) \times SO(2k)$ down to $U(N) \times U(M) \times U(k)$. Notice that unlike in the previous case, now the theories living on the D7 and on the D3 are both invariant under eight supercharges. The instanton moduli space for this model has been worked out in [10]. We refer the reader to this reference for details.

3.1 The D(-1)D7 system

We start again by describing the system in absence of D3-branes, whose contributions will be added later. The D7 brane theory is now half maximally supersymmetric and after reduction to four dimensions leads to a $\mathcal{N} = 2$ with gauge group $U(N)$ and two antisymmetric hypermultiplets.

Including k D(-1) instantons, the symmetry group of the theory becomes $U(N) \times U(k)$ with the $(B_{1,2}, D_{1,2})$ moduli transforming in the adjoint of the $U(k)$. The $B_{3,4}$ transform in the fundamental and anti fundamental representations of the gauge group, $\square + \overline{\square}$, and $D_{3,4}$ in the antisymmetric and its complex conjugate $\square + \overline{\square}$. Finally the D(-1)D7 open strings and their associated ADHM constraint transform in the $(\square, \overline{\square})$ and (\square, \square) bifundamental representations respectively of $U(N) \times U(k)$.

The k -instanton partition function $Z_{D7,k}$ can be written again as (2.5), now with integrand

$$w_{D7}(\chi, a) = \frac{c_k}{k!} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \prod_{\ell=1}^2 \prod_{i < j}^k \frac{(\chi_{ij,-}^2 - s_\ell^2)(\chi_{ij,+}^2 - s_{\ell+2}^2)}{(\chi_{ij,-}^2 - \epsilon_\ell^2)(\chi_{ij,+}^2 - \epsilon_{\ell+2}^2)} \\ \times \prod_{\ell=1}^2 \prod_{i=1}^k \frac{1}{(4\chi_i^2 - \epsilon_{\ell+2}^2)} \prod_{i=1}^k \prod_{u=1}^N (\chi_i - a_u) \quad (3.29)$$

$c_k = (-)^k$ and $\chi_{ij,\pm} = \chi_i \pm \chi_j$. The eight dimensional prepotential and chiral correlators are defined as before via (2.10) and (2.14).

3.2 The coupling on the D3-brane probes

Now let us consider the theory on a D3-brane probe. The theory on the probe is $\mathcal{N} = 2$ supersymmetric with gauge group $U(M)$. The matter content is given by two hypermultiplets transforming in the antisymmetric representation and N chiral multiplets transforming in the fundamental representation of $U(M)$.

The beta function coefficient becomes ⁶

$$b_{U(M)} = 4 - N \quad (3.30)$$

As before the case $N = 4$ is special in the sense that the four dimensional gauge theory is conformal.

Now let us consider the contributions of D(-1)-instantons. The instanton partition function $Z_{D3,k}$ can be written again as (2.16) but now with w_{D7} given by (3.29) and [10]

$$w_{D3}(\chi, b) = \prod_{i=1}^k \prod_{m=1}^M \frac{\left((\chi_i + b_m)^2 - \frac{(\epsilon_3 - \epsilon_4)^2}{4} \right)}{\left((\chi_i - b_m)^2 - \frac{(\epsilon_1 + \epsilon_2)^2}{4} \right)} \quad (3.31)$$

The four dimensional prepotential and gauge couplings are defined as before by (2.18) and (2.19).

Now we would like to relate the gauge coupling on the probe to the chiral ring of the D7 brane theory. Trying to follow the same strategy as before,

⁶The index of $U(N)$ representations are $T(\text{adj}) = N$, $T(\square) = \frac{N-2}{2}$, $T(\square) = \frac{1}{2}$.

we immediately run into problems because the function w_{D3} does not fall to one at $\epsilon_\ell \rightarrow 0$. This can be cured by choosing the regular D3-branes symmetrically distributed around the origin

$$b_m = \begin{cases} (b_n, -b_n) & \text{for } M \text{ even} \\ (b_n, -b_n, 0) & \text{for } M \text{ odd} \end{cases} \quad n = 1, \dots, [\frac{M}{2}] \quad (3.32)$$

For this choice one finds

$$w_{D3}(\chi, b) = 1 + 2\epsilon_3\epsilon_4 \sum_{J=0}^{\infty} \sum_{n=1}^{[M/2]} \frac{\text{tr} \chi^J}{b_n^{J+2}} (J+1) + \dots \quad (3.33)$$

Following the same steps as in the $SO(2N)$ case one is left with

$$2\pi i \tau_n^{\text{inst}} = 2 \left\langle \ln \det \left(1 - \frac{\Phi}{b_n} \right) \right\rangle_{\text{inst}} \quad (3.34)$$

As before, the elementary $SU(2)$ probe, i.e. $M = 2$, captures the full information about the axion-dilaton background. It is important to notice that for the choice (3.32) only correlators involving even powers of Φ are non-vanishing. Moreover, by an explicit evaluation of the eight-dimensional chiral correlators, one finds that the results are just half of those for the $SO(2N)$ case leading to the same answer for the gauge coupling τ_m of the elementary $SU(2)$ probe. This is not surprising since the two probe theories share their $SU(2)$ content since the antisymmetric representation of $SU(2)$ is the trivial in this case.

4 The axion dilaton profile

The operator $\mathcal{O}_\tau(b, q, \Phi)$ entering in the eight dimensional chiral correlator found before can be alternatively derived from string disk diagrams describing the rate of emission of a dilaton-axion field from D7, O7 and D(-1) sources. This has been done in [4] for the $SO(8)$ case but it holds in the more general settings under consideration here. In this section we review these results and comment on the generalization to the orbifold case.

The axion-dilaton profile emitted by a D-brane source is computed by a disk involving a closed string vertex for this field and any number of open string insertions. For concreteness let us focus on the dilaton field with string vertex

$$V_\phi = \delta\phi(p) c\tilde{c} e^{-\varphi-\tilde{\varphi}} \Psi^M \tilde{\Psi}_M e^{ip \cdot X} \quad (4.35)$$

where $\delta\phi(p)$ is the field polarization, Ψ, X fermionic and bosonic fields and c, φ ghosts and superghosts. As usual the tilde indicates fields of opposite handedness. We consider a dilaton profile along the two dimensional plane transverse to the D7 branes and therefore we take the momentum p of the closed string vertex along this plane. Denoting by a and χ the position matrices of the D7 and D(-1) instantons respectively along this plane, the effective coupling of the dilaton to the D(-1), D7 sources are given by the disk amplitudes

$$\begin{aligned} \left\langle V_\phi e^{-\frac{ia}{2\pi} \int d\tau \partial_\tau X} \right\rangle_{\text{disk}, D7} &\sim \delta\phi \text{tr} e^{ip \cdot a} \\ \left\langle V_\phi e^{-\frac{i\chi}{2\pi} \int d\tau \partial_\tau X} \right\rangle_{\text{disk}, D(-1)} &\sim \delta\phi \text{tr} e^{ip \cdot \chi} \end{aligned} \quad (4.36)$$

In presence of an O7 plane one finds an extra contribution $\langle V_\phi \rangle_{O7} \sim -8\delta\phi$ and the amplitudes are projected onto their even parts under reflections i.e. the matrices a and χ are projected into the adjoint of $SO(2N)$ and $SO(K)$ groups respectively. Similar couplings are found for the axion field C_0 , that results in the replacement $\delta\phi \rightarrow \delta\tau$ in (4.36) with $\tau = e^\phi + iC_0$ the axion-dilaton field. After Fourier transforming one can write the D7 contribution to the axion-dilaton effective action as

$$\delta S_{\text{eff}, D7} = -2\pi i \int d^8x \text{tr} e^{a \cdot \partial_z} \delta\tau(z) + \text{h.c.} \quad (4.37)$$

On the other hand the coupling to the D-instanton modifies the instanton moduli space by

$$\delta S_{\text{mod}} = -2\pi i \text{tr} e^{\chi \cdot \partial_z} \delta\tau(z) + \text{h.c.} \quad (4.38)$$

The contribution of this term to the effective action can be computed by the standard localization formulae

$$S_{\text{eff}, D(-1)} = \int d^8x d^8\theta \mathcal{F}_{D7}(M, T) \quad (4.39)$$

with $\mathcal{F}_{D7}(M, T)$ the eight dimensional prepotential

$$\mathcal{F}_{D7}(a, \delta\tau) = - \lim_{\epsilon_\ell \rightarrow 0} \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \ln Z_{D7}(a, \delta\tau) \quad (4.40)$$

given in terms of the multi instanton partition function

$$Z_{D7}(a, \delta\tau) = 1 + \sum_{K=1}^{\infty} q^K \int d\mathcal{M}_K e^{-S_{\text{mod}}(\mathcal{M}_K, a) - \delta S_{\text{mod}}(\delta\tau)} \quad (4.41)$$

Finally

$$\begin{aligned} M &= \Phi + \dots \\ T &= \tau + \dots + \theta^8 \partial^4 \bar{\tau} \end{aligned} \quad (4.42)$$

are the superfields containing the chiral field Φ on the D7-brane and the axion-dilaton field respectively. To linear order in δT one finds

$$\begin{aligned} \delta \mathcal{F}_{D7}(M, \delta T) &= -2\pi i \sum_{J=0}^{\infty} \frac{\partial^J \delta T}{J!} \lim_{\epsilon_{\ell} \rightarrow 0} \frac{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}{Z_{D7}} \\ &\quad \times \sum_K q^K \int d\mathcal{M}_K e^{-S_{\text{mod}}(\mathcal{M}_K, M)} \text{tr} \chi^J \\ &= -2\pi i \sum_{J=0}^{\infty} \frac{\partial^J \delta T}{(J+4)!} \langle \text{tr} \Phi^{J+4} \rangle_{\text{inst}} \end{aligned} \quad (4.43)$$

where in the second line we used (2.14) to relate the integrals in right hand side to the chiral correlators in the D7 gauge theory. Plugging this into (4.39) and using the highest component of T in (4.42) to soak the θ -integrals one finally finds

$$\delta S_{\text{eff}, D(-1)} = -2\pi i \int d^8 x \langle \text{tr} e^{\Phi \cdot \partial_z} \rangle_{\text{inst}} \delta \tau(z) + \text{h.c.} \quad (4.44)$$

Notice that (4.44) is nothing but the non-perturbative analog of the coupling $S_{\text{eff}, D7}$ found in (4.37) and therefore one can write the two together as

$$\begin{aligned} \delta S_{\text{eff}} &= \delta S_{\text{eff}, D7} + \delta S_{\text{eff}, D(-1)} \\ &= -2\pi i \int d^8 x \langle \text{tr} e^{\Phi \cdot \partial_z} \rangle_{D7} \delta \tau(z) + \text{h.c.} \end{aligned} \quad (4.45)$$

with

$$\langle \text{tr} \Phi^J \rangle_{D7} = \text{tr} a^J + \langle \text{tr} \Phi^J \rangle_{\text{inst}} \quad (4.46)$$

denoting the full chiral correlator in the eight dimensional theory with the first term accounting for its classical part. Taking into account the bulk kinetic term

$$S_{\text{bulk}} = \int d^{10} x \bar{\tau} \partial \bar{\partial} \tau \quad (4.47)$$

and varying with respect to $\bar{\tau}$ one finds the equation of motion

$$\partial_z \bar{\partial}_z \tau = 2\pi i \left[\sum_{J=0}^{\infty} \frac{1}{J!} \partial_z^J \delta^{(2)}(z) \langle \text{tr } \Phi^J \rangle_{D7} - 8\delta^{(2)}(z) \right] \quad (4.48)$$

that is solved by

$$2\pi i \tau(z) = 2\pi i \tau_0 + \langle \ln \det(z - \Phi) \rangle_{D7} - 8 \ln z \quad (4.49)$$

The harmonic part in (4.49) has been fixed by matching the one-loop result on the two sides of the equation. The result (4.49) agrees with the gauge coupling found from the direct multi-instanton computation (2.22) on the D3-brane probe with $z \rightarrow b_m$. The generalization to the orbifold case is straightforward. We notice that the orbifold acts as a projection on the open string degrees of freedom and therefore all formulae in this section still apply with the only difference that now the moduli space integrals defining the chiral correlators run over the \mathbb{Z}_2 -invariant moduli subspace.

Acknowledgments

We thank M.Billò, M.L.Frau, A.Lerda and L.Martucci for useful discussions. This work was partially supported by the ERC Advanced Grant n.226455 “*Superfields*”, by the Italian MIUR-PRIN contract 20075ATT78, by the NATO grant PST.CLG.978785.

A Eight dimensional correlators

In this appendix we collect the results for the eight-dimensional correlators. We start by considering the case of D7-branes in flat ten-dimensional space realizing an $SO(2N)$ gauge theory in eight-dimensions. The correlators follow from explicit evaluation of the integrals in (2.14). Up to order q^3 one finds

the non-trivial correlators [6] ⁷

$$\begin{aligned}
\langle \text{tr} \Phi^4 \rangle_{\text{inst}} &= -12 q \sqrt{A_N} - \frac{3 q^2 A_{N-2}}{8} - \frac{3}{16} q^3 A_{N-4} \sqrt{A_N} \\
\langle \text{tr} \Phi^6 \rangle_{\text{inst}} &= \frac{45 q^2 A_{N-1}}{8} + \frac{15}{8} q^3 A_{N-3} \sqrt{A_N} \\
\langle \text{tr} \Phi^8 \rangle_{\text{inst}} &= -\frac{105}{4} q^2 A_N - \frac{35}{4} q^3 A_{N-2} \sqrt{A_N} \\
\langle \text{tr} \Phi^{10} \rangle_{\text{inst}} &= \frac{105}{4} q^3 A_{N-1} \sqrt{A_N} \\
\langle \text{tr} \Phi^{12} \rangle_{\text{inst}} &= -\frac{495}{8} q^3 A_N^{3/2}
\end{aligned} \tag{A.50}$$

with A_m , $m = 1, \dots, N$ given in (2.24).

In the case of $U(N)$ plugging (3.29) into (2.14) one finds that the results for the correlators are half of those in (A.50).

⁷With respect to [6] we perform the rescaling $q \rightarrow -\frac{q}{16}$.

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